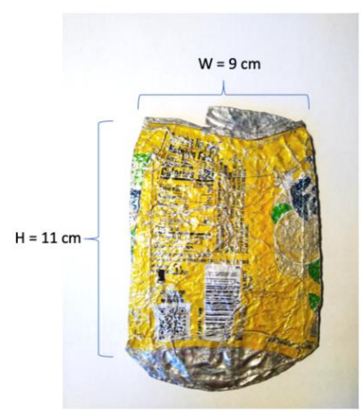
Flat Cans

Have you ever had the experience of walking through a parking lot or along a road and finding a squashed aluminum can, thoroughly flattened, having been run over a few dozen times? Maybe you have even picked one up, looking for the nearest recycling bin. Well, it was a sunny day in Seattle, and I was walking from one place to another when I had just such an experience with an exceptionally well-crushed can. I picked it up and found myself thinking about the transformation of the container from cylinder to rectangle. I had been working with students recently to help them develop their understanding of the way in which volumes for regular right prisms, like a can or a shoebox, could be calculated. All you need to know is the height of the prism and the area of the base, whether that base is a rectangle, a circle, a triangle, or what have you. Multiplying the area of the base by the height gives you number of times the bottom layer is repeated, in a sense, and therefore the volume of the object.

Looking at my flat can, I was delighted by the realization that the dimensions of the rectangle I was holding could be used to infer the dimensions of the original cylinder. The approximate height of the cylindrical can was still easily seen in the flattened version, and the width of the more or less two-dimensional rectangle had come from the flattening of the three-dimensional cylinder. Sure, being run over and over is going to cause some deformation; but within reason, the width of the rectangle—when doubled to account for both sides—had to be a good approximation for the circumference of the original cylinder. And of course if you know the circumference of a circle, you can calculate the radius; and with that you can use the usual formula for the area of a circle to calculate the area of the base. So, with the height and now the area of the base of my right circular prism, aka can, the original volume could be estimated!

Here is how it worked out for my can (where H = height, W = width of the flat can, C and r are the circumference and radius of the original unflattened can, and V is its volume):



H = 11cm, W = 9 cm, and $C = 2W$, so C = 18 cm.
We also know that $C = 2\pi r$, so 18 cm $= 2\pi r$, and so $r \approx 2.9$ cm.

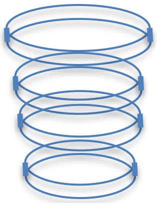
The volume of the original can is given by the product of the area of the base and the height ($V = \pi r^2 \times H$), so
$V = \pi(2.9\text{ cm})^2 \times (11\text{cm}) \approx 2.90\text{cm}^3$. Using the conversion factor, 1 cm^3 = 0.034 fluid ounces, we estimate that the original can would have held

9.9, or about 10, ounces. I could not be sure, but this looked like it was a 12-ounce can, so this estimate is a bit low. This seems to happen often, and I attribute it to the small creases the can gets as it is crushed and repeatedly run over.

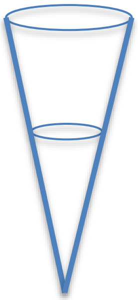
I figured that after a bit of cleaning, rather than the recycling bin, this can was headed for my classroom, where I knew it would give the learners a chance to exercise and perhaps extend our recent work with area and volume. I began looking for more flat cans, perhaps with different original volumes, hoping to enable some small-group work, different answers, and different approaches to the solution. For some of my students, though, this would not be enough, and just a few minutes later I found a flat cup!

The flat cup, because it was not originally a right prism, presents some interesting challenges if you want to determine the volume of the preflattened object. As with most cups, the radius at the base is less than that at the top. You can apply the same reasoning used on the can, but the width is a constantly changing measure depending on how far up the cup you take the measurement. What would my learners do to come up with a defensible estimate of the cup's original volume? What would *you* do?

It would not be unreasonable, as a first pass, to take the measure of the flat cup's width halfway up from the bottom to the top, let that width stand for half of the circumference, and then approximate the volume of the whole cup by treating it as a right cylinder with the radius derived from that measurement.



But could you do better than that? What if you took measurements of width at several points from the base to the top and calculated the volume as the sum of a series of short and ever-widening right cylinders stacked one on the next? As you increase the number of these cylinders (while decreasing their heights, of course), you will get a more and more accurate estimate of the cup's volume. And this sets the stage for a discussion of some of the fundamental notions of calculus.



Alternatively, you could imagine the cup as the top section of a cone. To find the volume of a cone, you need to know only its height and the radius of its base, which for us would be the top of the cup. How would you get a reasonable estimate for the height of this imaginary cone? What would *you* do?

Once you are alert for and interested in them, you will notice all kinds of flattened things along the side of the road. Plastic water bottles are all too common. With their more complex original three-dimensional shapes, estimating volumes from their flattened remains presents some real challenges and

probably because my own mathematical skills are not all they could be, I lose interest. Some of these shapes take me beyond my own zone of proximal development, as Vygotsky might say, and this is a good reminder of the importance of finding challenges that will engage our students rather than simply frustrate or bore them.
Nearing my destination, I came across one more flattened object that made me think. It was a flat candy box. It must have been in the road for some time because it was so scuffed and worn that although it was still pretty much rectangular as a two-dimensional object, it was no longer possible to distinguish what had once been the sides from the front or back. So, is it possible to determine the volume of the original rectangular prism from this flattened remnant? Is it?

I had the height of the original rectangular prism, but the area of the base depended on two dimensions, the width and the depth; and those could no longer be detected in this thoroughly worn object. I know that the most efficient use of the materials in the package would have dictated that all four sides have the same dimensions so that the package was a square prism. That is how you get the maximum enclosed volume for any given amount of material used to make the box. But boxes made this way are very rare. Why is that? Is it the need for a big front panel with a splashy image and logo, maximizing advertising space while reducing the volume that has to be filled with actual product? What would my students think?

Questions like this bring the lights of other subjects into the math classroom. When you open the door to allow social issues, product design, engineering, and the like into the room, you make it easier for mathematics to make its way out of the strict confines of math classrooms and you increase the chances that math will be allowed to stay in the cognitive lives of your learners.

Lesson Plan

Learn more about implementing Flat Cans in your classroom by exploring the Illuminations lesson here! Then, share your experiences using Math Sightings on social media with the hashtag #MathSightings.